$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta, \text{ or}\\ \sin^2\alpha &\geq (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta.\end{aligned}$$

Adding $\cos^2\alpha$ to both sides of the above inequality we obtain

$$\begin{aligned}1 &\geq \cos^2\alpha-\cos^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta=\cos^2\alpha(1-\sin^2\beta)+\sin^2\alpha\cos^2\beta\\ 1 &\geq \cos^2\alpha\cos^2\beta+\sin^2\alpha\cos^2\beta=(\cos^2\beta)(\cos^2\alpha+\sin^2\alpha)\\ 1 &\geq \cos^2\beta, \text{ and this proves the conjecture.}\end{aligned}$$

Also solved by Michael Brozinsky, Central Islip, NY; Jerry Chu (student, Saint George's School), Spokane, WA; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro GA, and the proposer.

- **5345:** *Proposed by Arkady Alt, San Jose, CA*

  Let $a, b > 0$. Prove that for any x, y the following inequality holds

  $$|a\cos x+b\cos y| \leq \sqrt{a^2+b^2+2ab\cos(x+y)},$$

  and find when equality occurs.

Solution 1 by Michael Brozinsky, Central Islip, NY

Since $\sqrt{u^2}=|u|$, the left hand side of the given inequality can be written as

$$a^2\cos^2 x+2ab\cos x\cos y+b^2\cos^2 y,$$

and so using the identities $\sin^2 u = 1-\cos^2 u$ and $\cos(x+y)=\cos x\cos y-\sin x\sin y$, it must be shown that

$$a^2\sin^2 x+b^2\sin^2 y \geq 2ab\sin x\sin y.$$

This is true from the AM-GM inequality, with equality if, and only if, $a\sin x = b\sin y$.

Solution 2 by Paul M. Harms, North Newton, KS

Since each side of the inequality is a nonnegative number, the inequality holds if the square of the left side is less than or equal to the square of the right side. We need to show that

$$(a\cos x+b\sin y)^2=a^2\cos^2 x+2ab\cos x\cos y+b^2\cos^2 y \leq a^2+b^2+2ab\cos(x+y).$$

The last inequality is equivalent to

$$\begin{aligned} 0 &\le a^2\left(1-\cos^2 x\right)+b^2\left(1-\cos^2 y\right)+2ab\left(\cos(x+y)-\cos x\cos y\right) \\ &= a^2\sin^2 x+b^2\sin^2 y+2ab((\cos x\cos y-\sin x\sin y)-\cos x\cos y) \\ &= (a\sin x-b\sin y)^2. \end{aligned}$$

Clearly, $0 \le (a\sin x - b\sin y)^2$ so the problem inequality holds. Equality will hold when $a\sin x = b\sin y$ or $\dfrac{a}{b}=\dfrac{\sin x}{\sin y}$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN; Nicusor Zlota, "Traian Vula" Technical College, Focsani, Romania, and the proposer.

- **5346:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Show that in any triangle ABC, with the usual notations, the following hold,

$$\frac{h_b+h_c}{h_a}r_a^2+\frac{h_c+h_a}{h_b}r_b^2+\frac{h_a+h_b}{h_c}r_c^2 \ \ge\ 2s^2,$$

where r_a is the excircle tangent to side a of the triangle and s is the triangle's semiperimeter.

Solution 1 by Moti Levy, Rehovot, Israel

From geometry of the triangle:

$$h_a=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_c}},\ h_b=\frac{2}{\dfrac{1}{r_a}+\dfrac{1}{r_c}}\ ,\ h_c=\frac{2}{\dfrac{1}{r_b}+\dfrac{1}{r_a}}. \qquad (1)$$

Solving (1) for r_a, r_b and r_c, we get

$$r_a \ = \ \frac{h_ah_bh_c}{h_ah_b+h_ah_c-h_bh_c}$$